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First-passage probability of a random walk on a disordered one-dimensional lattice

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Abstract. A method is presented which yields the exact solution of the generating function for the first-passage probability and related observables, for a one-dimensional random walk with random hopping rates on each site. A geometrical and statistical-mechanical interpretation of the problem is suggested and is used for obtaining the asymptotic properties of the occupation and first-passage probabilities. The class of values with the property of self-averaging is found.

1. Introduction

Anomalous diffusion in random systems has received wide attention in the last decade [1–3]. In spite of considerable progress, many important problems are still open. One of the models, intensively studied in the literature, is a one-dimensional discrete-time random walk on the random lattice [3–10]. A few attempts to describe the statistics of the first-passage time in this model have recently been reported. In [7–9] an explicit expression for the mean first-passage time was presented, while in [10] much information about first-passage time distribution, including an asymptotic expression for the first-passage time probability, was obtained. The object of main importance in these works is the generating function of the first-passage probability.

This paper is a study of the statistical properties of the first-passage time on a disordered one-dimensional lattice. The key step in our formulation consists of a summation over trajectories of a random walker and an investigation of the properties of the set of all the trajectories. We consider only relatively simple cases, because our main purpose here is to demonstrate our method.

The outline of the paper is as follows. In section 2, we describe the model and derive a system of equations for the generating function. In section 3, we find an exact solution of this system and present explicit expressions for the generating function of first-passage probability in terms of the basic transition probabilities. In section 4, it is shown that the properties of random walks are essentially determined by certain combinatorical characteristics of the one-dimensional lattice, namely the number of trajectories passing every site of the lattice a given number of times. The explicit expression for this number, found in section 4, yields a new representation of the generating function, which has the form of a partition function. The saddle-point approximation, applied to this partition function, provides asymptotic estimates for the time dependence of the first-passage probability. In section 5, the partition-function representation is used to prove the property of self-averaging of a class of

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observables. In appendix A, the system of equations of section 2 is used to derive an explicit expression for the first and second moments of first-passage time in terms of the basic transition probabilities. In appendix B, we list some different cases, when the generating function may be presented explicitly.

2. Equations for generating functions

Consider a random walk which jumps on a discrete one-dimensional lattice at integer times from site k to site $k-1$ with probability L_k, or to site $k+1$ with probability $R_k = 1 - L_k$. The hopping probabilities L_k are chosen independently from site to site. We will be mostly interested in a consideration of the problems with a given choice of $\{L_k\}$, rather then in averaging over different possible choices.

Let $W_n(t)$ be the probability for a random walker to start from point 0 and arrive at point $n > 0$ (the lattice is allowed to have negative as well as positive sites) for the first time in t steps (the first-passage probability). It is helpful to introduce the corresponding generating function [11, 12]

$$p_n(x) = \sum_{t=0}^{\infty} W_n(t) x^t. \tag{2.1}$$

From the generating function, $W_n(t)$ may be obtained as

$$W_n(t) = \frac{1}{t!} \frac{\mathrm{d}^t}{\mathrm{d}x^t} p_n(x) \bigg|_{x=0} \tag{2.2}$$

while the kth moment of first-passage time is

$$\overline{t_n^k} = \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^k p_n(x) \bigg|_{x=1}. \tag{2.3}$$

Thus $p_n(x)$ contains the same information as $W_n(t)$.

If we denote $l_k = xL_k$ and $r_k = xR_k$ then the rule for calculating p_n (we will drop the argument x) is the same as for calculating a Feynman amplitude: one sums over all paths between 0 and n (coming to n only once at the last step) containing all possible numbers of steps, and the contribution of each path is the product of corresponding l_k or r_k for every step.

Let us also define $b_n(x)$, the generating function for the *occupation* probability to start from site n to the left and return back to n without visiting site $n+1$ on the way. For calculating $b_n(x)$ one sums over all the paths from n to n 'through the left', using the same rules for the contribution of every path as for $p_n(x)$. On figure 1 we give the examples of the paths contributing to $p_n(x)$ and $b_n(x)$.

Using these rules one obtains the following important relations:

$$p_n = p_{n-1} b_{n-1} r_{n-1} \tag{2.4}$$

$$b_n = 1 + l_n b_{n-1} r_{n-1} b_n \tag{2.5}$$

where 1 in right-hand side of (2.5) corresponds to the term $t = 0$ in the definition of b_n (see (2.1)).

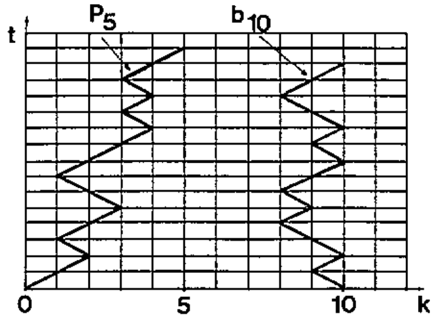

Figure 1. Typical paths, contributing to $p_5(x)$ and $b_{10}(x)$.

By iterating (2.4) one gets

$$p_n = \prod_{k=0}^{n-1} b_k r_k. \tag{2.6}$$

This formula expresses $p_n(x)$ through $\{b_k\}_{k=0,\dots,n-1}$ while (2.5) gives the closed system of equations for the bs, which it is convenient to express in the form†

$$b_n = \frac{1}{1 - l_n r_{n-1} b_{n-1}}. \tag{2.7}$$

The solution of this system, presented in the next section, yields by (2.6) an explicit expression for the generating function.

It is obvious from (2.3) and (2.6) that in order to find $\overline{t_n^i}$ it is sufficient to know b_k near $x = 1$ to order y^i, where $y = 1 - x$. In appendix A we shall find the explicit expression for b_k to order y^2, which will allow us to determine the first two moments of the first-passage time.

3. Exact solutions for generating functions and probabilities

In this section we will find the exact solution of (2.7). For later convenience, let us denote $z = x^2$, $c_0 = L_1 R_0 b_0$, and $c_k = L_{k+1} R_k$ and $k = 1, 2, \dots$. Then from (2.7) one has

$$b_1 = \frac{1}{1 - zc_0} \tag{3.1}$$

$$b_k = \frac{1}{1 - zc_{k-1} b_{k-1}} \qquad k = 2, 3, \dots. \tag{3.2}$$

It follows from (3.2), that if one seeks b_k in the form

$$b_k = N_k / D_k \tag{3.3}$$

† For a more general case, when the walk at every integer moment is allowed not only to hop, but also to stay at the same site with non-zero probability, the system of equations, equivalent to (2.6) and (2.7), was presented in [8].

where N_k and D_k are polynomials of z, then $N_k = D_{k-1}$. This means that the product $\prod_1^n b_k$, which contributes to p_{n+1}, is equal to D_n^{-1}. The recurrence relation for D_k following from (3.2) is

$$D_k = D_{k-1} - z c_{k-1} D_{k-2}. \tag{3.4}$$

If one seeks D_k as

$$D_k = \sum_{m \geqslant 0} b_k^{(m)} z^m \tag{3.5}$$

then the recurrence relations for coefficients $b_k^{(m)}$ are

$$\begin{aligned} b_k^{(0)} &= b_{k-1}^{(0)} \\ b_k^{(m)} &= b_{k-1}^{(m)} - c_{k-1} b_{k-2}^{(m-1)} \qquad m = 1, 2, \ldots. \end{aligned} \tag{3.6}$$

These relations, together with the initial conditions, following from (3.1), lead to the explicit solution, which may be easily proved by induction:

$$\begin{aligned} b_k^{(0)} &= 1 \\ b_k^{(m)} &= (-1)^m \sum_{j_1=2m-2}^{k-1} c_{j_1} \sum_{j_2=2m-4}^{j_1-2} c_{j_2} \cdots \sum_{j_m=0}^{j_{m-1}-2} c_{j_m} \qquad m = 1, 2, \ldots, m_k \\ b_k^{(m)} &= 0 \qquad m > m_k \end{aligned} \tag{3.7}$$

where m_k is the integer part of $\frac{1}{2}(k+1)$:

$$m_k = [\tfrac{1}{2}(k+1)] \equiv \begin{cases} j & k = 2j \\ j+1 & k = 2j+1. \end{cases} \tag{3.8}$$

For example (3.7) yields

$$\begin{aligned} D_6 = 1 &- z(c_0 + \cdots + c_5) + z^2(c_2 c_0 + \cdots + c_5 c_0 + c_3 c_1 + c_4 c_1 + c_5 c_1 + c_4 c_2 + c_5 c_2 + c_5 c_3) \\ &- z^3(c_4 c_2 c_0 + c_5 c_2 c_0 + c_5 c_3 c_0 + c_5 c_3 c_1). \end{aligned}$$

Let us restrict ourselves now to the case of the half-infinite lattice, having only sites with $n \geqslant 0$. Then $b_0 = 1$, $R_0 = 1$ and $c_0 = L_1$. Let us denote

$$\Re_n = \prod_{k=1}^{n} R_k \tag{3.9}$$

then one finally gets

$$p_{n+1} = x^{n+1} \Re_n \prod_{k=1}^{n} b_k = x^{n+1} \Re_n \frac{1}{\sum_{m=0}^{m_n} b_n^{(m)} z^m} \tag{3.10}$$

with $b_n^{(m)}$ and m_n from (3.7) and (3.8). We note that all trajectories, contributing to the generating function, must nesessarily include at least one jump from every site $k \leqslant n$ to the site $k+1$, so p_{n+1} must contain a trivial overall factor $\prod_{k=0}^{n} r_k = x^{n+1}\Re_n$. This factor is explicitly separated out in (3.10), while the non-trivial part of p_{n+1} is contained in the factor $\prod_{k=1}^{n} b_k$.

By use of known formulae [13] for the inverse series one obtains

$$p_{n+1} = x^{n+1}\Re_n \sum_{m=0}^{\infty} d_n^{(m)} z^m = \Re_n \sum_{m=0}^{\infty} d_n^{(m)} x^{2m+n+1} \tag{3.11}$$

where

$$d_n^{(0)} = 1 \qquad d_n^{(1)} = -b_n^{(1)}$$

$$d_n^{(m)} = (-1)^m \det \begin{vmatrix} b_n^{(1)} & 1 & 0 & 0 & \dots & 0 & 0 \\ b_n^{(2)} & b_n^{(1)} & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ b_n^{(m-1)} & b_n^{(m-2)} & b_n^{(m-3)} & b_n^{(m-4)} & \dots & b_n^{(1)} & 1 \\ b_n^{(m)} & b_n^{(m-1)} & b_n^{(m-2)} & b_n^{(m-3)} & \dots & b_n^{(2)} & b_n^{(1)} \end{vmatrix} \qquad m \geqslant 2. \tag{3.12}$$

Using (2.2) the first-passage probability is derived:

$$W_{n+1}(t) = \begin{cases} 0 & t < n+1 \text{ or } t-(n+1) = 2m+1 \\ \Re_n d_n^{(m)} & t-(n+1) = 2m \end{cases} \tag{3.13}$$

where $m = 0, 1, 2, \dots$.

Equations (3.10)–(3.13) yield an explicit expression for the first-passage probability and its generating function in terms of the basic transition probabilities for a particular realization of the random variables. Some other cases, when the generating function may be presented explicitly, are listed in appendix B.

It is instructive to illustrate the work of (3.12) and (3.13) by considering the simple cases of $m = 1, 2$. In the case of $m = 1$ or $t = n+1+2$ the walker makes one jump to the left on his way from 0 to $n+1$. Since this 'back jump' may be made from every site k, $1 \leqslant k \leqslant n$, and occurs with a probability $L_k R_{k-1} = c_{k-1}$, it is clear, that the probability of a path with one ($m = 1$) back jump is equal to $\Re_n \sum_{k=0}^{n-1} c_k$ in agreement with (3.13). Similarly, in the case of $m = 2$ the walker makes two back jumps, and the corresponding factor in the probability is the product of corresponding cs. Then taking into account that there is one possibility of making a back jump from sites k, $k' \geqslant k$ if $k' \neq k+1$ and two such possibilities if $k' = k+1$ (see figure 2), one arrives at the result (3.13) for $W_{n+1}(t = n+1+4)$, where by (3.12)

$$d_n^{(2)} = \det \begin{vmatrix} b_n^{(1)} & 1 \\ b_n^{(2)} & b_n^{(1)} \end{vmatrix}.$$

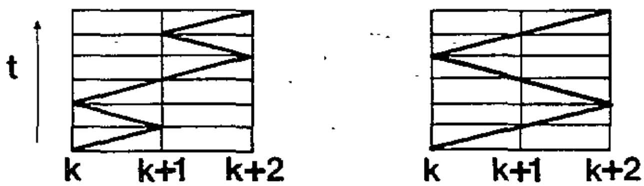

Figure 2. Two possibilities of making back jumps from sites $k+1$ and $k+2$.

Expression (3.10), together with (2.3), may be used for obtaining different moments of first-passage time. To this end the important relation is

$$\sum_{m=0}^{m_n} b_n^{(m)} = \Re_n \tag{3.14}$$

following from $p_{n+1}|_{x=1} = 1$.

4. Partition-function representation and an estimation of probabilities

4.1. The partition-function representation

In this section we present a new representation of the generating function, which allows us to obtain some asymptotic estimations for the time dependence of the first-passage probability.

Let us consider the simplest case of a half-infinite lattice with sites $n \geqslant 0$ and a walker starting from site $n = 0$. We denote the non-trivial part of p_{n+1} as $\tilde{p}_{n+1}$:

$$\tilde{p}_{n+1} \equiv \prod_{k=1}^{n} b_k = x^{-n-1}\Re_n^{-1} p_{n+1} \tag{4.1}$$

and expand $\tilde{p}_{n+1}$ in a Taylor series in the variables $\{c_k\}$:

$$\tilde{p}_{n+1} = \sum_{k_0,\ldots,k_{n-1}=0}^{\infty} z^{k_0+\cdots+k_{n-1}} c_0^{k_0} \ldots c_{n-1}^{k_{n-1}} N_{k_0,\ldots,k_{n-1}}. \tag{4.2}$$

For the coefficients $N_{k_0,\ldots,k_{n-1}}$ one has

$$\begin{aligned} N_{k_0,\ldots,k_{n-1}} &= \frac{1}{k_0!\cdots k_{n-1}!}\frac{\partial^{k_0}}{\partial c_0^{k_0}}\cdots\frac{\partial^{k_{n-1}}}{\partial c_{n-1}^{k_{n-1}}}\tilde{p}_{n+1}\bigg|_{\substack{z=1\\ c_0=\cdots=c_{n-1}=0}} \\ &= \frac{1}{k_0!\cdots k_{n-1}!}\frac{\partial^{k_0}}{\partial c_0^{k_0}}\cdots\frac{\partial^{k_{n-1}}}{\partial c_{n-1}^{k_{n-1}}}\prod_{k=1}^{n} b_k\bigg|_{\substack{z=1\\ c_0=\cdots=c_{n-1}=0}}. \end{aligned} \tag{4.3}$$

The physical interpretation of (4.2) is as follows: a given set of $\{k_i\}_{i=0,\ldots,n-1}$ corresponds to the paths with k_i left and right jumps between sites i and $i+1$, not counting the first necessary one from i to $i+1$, which is included in p_{n+1} (see (4.1)). Every such back jump gives a factor $c_i = L_{i+1}R_i$ to the probability or a factor $c_i z = c_i x^2$ to the generating function, $N_{k_0,\ldots,k_{n-1}}$ is the number of paths with given $\{k_i\}_{i=0,\ldots,n-1}$ and (4.2) is the sum over all possible choices of $\{k_i\}$.

The expansion (4.2) (and the corresponding expansion for p_{n+1}) provides us with a new view of the problem of random walking. One can consider c_i as a signal, acting on the random walker, which now plays the role of the black box, and $\tilde{p}_{n+1}$ as a response function. By (4.2) inside the black box the signal convolutes with coefficients N, giving response $\tilde{p}_{n+1}$. The signal c_i may be fixed or a member of a random ensemble, correlated or uncorrelated, but the nature of the problem is determined by the independent constants N, which play a fundamental role in the random-walking processes.

The constants N can be easily determined from the second line of (4.3). The key point is that the b_k do not depend on c_i, $i \geqslant k$, so one has the relation

$$\frac{\partial^{k_0}}{\partial c_0^{k_0}} \cdots \frac{\partial^{k_{n-1}}}{\partial c_{n-1}^{k_{n-1}}} \prod_{k=1}^{n} b_k \bigg|_{\substack{z=1 \\ c_0=\cdots=c_{n-1}=0}} = \frac{\partial^{k_0}}{\partial c_0^{k_0}} b_1 \bigg|_{\substack{z=1 \\ c_0=0}} \frac{\partial^{k_1}}{\partial c_1^{k_1}} b_2 \bigg|_{\substack{z=1 \\ c_1=0}} \cdots \frac{\partial^{k_{n-1}}}{\partial c_{n-1}^{k_{n-1}}} b_n \bigg|_{\substack{z=1 \\ c_{n-1}=0}}$$

$$= \prod_{i=0}^{n-1} \frac{\partial^{k_i}}{\partial c_i^{k_i}} b_{i+1} \bigg|_{\substack{z=1 \\ c_i=0}} \tag{4.4}$$

where each derivative acts on the whole expression on the right. Using (3.2) it is easy to prove by induction over j that

$$\prod_{i=n-j}^{n-1} \frac{\partial^{k_i}}{\partial c_i^{k_i}} b_{i+1} \bigg|_{\substack{z=1 \\ c_i=0}} = b_{n-j}^{k_{n-j}} \prod_{i=n-j}^{n-2} \frac{(k_i + k_{i+1})!}{k_{i+1}!} \cdot k_{n-1}!. \tag{4.5}$$

Setting $j = n$ and using (4.3), we finally obtain

$$N_{k_0,\dots,k_{n-1}} = \prod_{i=0}^{n-2} \frac{(k_i + k_{i+1})!}{k_i! k_{i+1}!} = \prod_{i=0}^{n-2} \binom{k_i + k_{i+1}}{k_i}. \tag{4.6}$$

Let us now define the new quantity $\widetilde{W}_{k_0,\dots,k_{n-1}}$ as

$$\widetilde{W}_{k_0,\dots,k_{n-1}} = c_0^{k_0} \dots c_{n-1}^{k_{n-1}} N_{k_0,\dots,k_{n-1}}. \tag{4.7}$$

$\Re_{n-1} \widetilde{W}_{k_0,\dots,k_{n-1}}$ is the occupation probability of reaching site n from 0 'from the left' (i.e. without visiting site $n+1$) by all the paths with given set $k_0, \dots, k_{n-1}$. By (4.2)

$$\tilde{p}_{n+1} = \sum_{k_0,\dots,k_{n-1}=0}^{\infty} z^k \widetilde{W}_{k_0,\dots,k_{n-1}} \tag{4.8}$$

where

$$k = k_0 + \cdots + k_{n-1} \tag{4.9}$$

while the first-passage probability $W_{n+1}(t)$ to reach site $n+1$ and occupation probability $\widetilde{W}_n(t)$ to reach site n from the left satisfy the relation

$$\Re_n^{-1} W_{n+1} \bigg|_{t=n+1+2k} = \Re_{n-1}^{-1} \widetilde{W}_n \bigg|_{t=n+2k} = \sum_{k_0+\cdots+k_{n-1}=k} \widetilde{W}_{k_0,\dots,k_{n-1}}. \tag{4.10}$$

It is interesting to note that (4.8) and (4.10) relate the problem of a random walk to the statistical mechanics of a system of positive integer coordinates $k_0, \dots, k_{n-1}$ with Hamiltonian $-\log \widetilde{W}_{k_0,\dots,k_{n-1}}$. By (4.10), $\Re_n^{-1} W_{n+1}$ and $\Re_{n-1}^{-1} \widetilde{W}_n$ are the canonical partition functions of this system with given k, while by (4.8) $\tilde{p}_{n+1}$ is the grand canonical partition function with chemical potential $\mu = -\log z$. Thus we can use the methods developed in statistical mechanics for the evaluation of generating functions and corresponding probabilities.

4.2. The estimation of probabilities

As an example of an application of this formalism and the utilization of the methods of statistical mechanics, we evaluate the asymptotic time dependence of the sum in (4.10), which determines the probabilities W_{n+1} and $\widetilde{W}_n$. By (4.6) one has

$$\widetilde{W}_{k_0,\dots,k_{n-1}} = L_1^{k_0} \prod_{i=1}^{n-1} \frac{(k_{i-1}+k_i)!}{k_{i-1}!k_i!} (R_i L_{i+1})^{k_i}. \tag{4.11}$$

When all k_i are large, we use the Stirling formula $x! \simeq \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x}$, and rewrite (4.11) in the form

$$\widetilde{W}_{k_0,\dots,k_{n-1}} = \sqrt{(2\pi)^{-n+1} \prod_{i=1}^{n-1} \frac{k_{i-1}+k_i}{k_{i-1}k_i} e^{S(\boldsymbol{k})}} \tag{4.12}$$

where $\boldsymbol{k} = (k_0, \dots, k_{n-1})$ and

$$\begin{aligned} S(\boldsymbol{k}) = k_0 \log L_1 + \sum_{i=1}^{n-1} ((k_{i-1}+k_i)\log(k_{i-1}+k_i) - k_{i-1}\log k_{i-1} \\ - k_i \log k_i + k_i \log(R_i L_{i+1})). \end{aligned} \tag{4.13}$$

Now, denoting the sum in (4.10) as A and approximating it by an integral, one has

$$\begin{aligned} A &\equiv \sum_{k_0+\dots+k_{n-1}=k} \widetilde{W}_{k_0,\dots,k_{n-1}} \\ &\cong \sqrt{(2\pi)^{-n+1}} \int_{k_i \geqslant 0} \prod_{i=0}^{n-1} dk_i \, \delta\left(\sum_{i=0}^{n-1} k_i - k\right) \sqrt{\prod_{i=1}^{n-1} \frac{k_{i-1}+k_i}{k_{i-1}k_i} e^{S(\boldsymbol{k})}}. \end{aligned} \tag{4.14}$$

From (4.13) it is clear that $S(\boldsymbol{k})$ is a homogeneous function:

$$S(\gamma \boldsymbol{k}) = \gamma S(\boldsymbol{k}). \tag{4.15}$$

Thus introducing in (4.14) the new variables $\alpha_i = k_i/k$, one has

$$A = \left(\frac{k}{2\pi}\right)^{(n-1)/2} \int_{\alpha_i \geqslant 0} \prod_{i=0}^{n-1} d\alpha_i \, \delta\left(\sum_{i=0}^{n-1} \alpha_i - 1\right) \sqrt{\prod_{i=1}^{n-1} \frac{\alpha_{i-1}+\alpha_i}{\alpha_{i-1}\alpha_i} e^{kS(\boldsymbol{\alpha})}}. \tag{4.16}$$

In the statistical-mechanical interpretation just given k now clearly plays the role of the inverse temperature. For large k (small temperature), the only non-negligible contribution to A comes from the vicinity of the maximum of $S(\boldsymbol{\alpha})$ on the plane $\sum_{i=0}^{n-1} \alpha_i = 1$, which plays the role of the ground state. The position $\boldsymbol{\alpha}^{(m)}$ of this maximum is determined by the equations

$$\frac{\partial S}{\partial \alpha_0^{(m)}} = \frac{\partial S}{\partial \alpha_1^{(m)}} = \dots = \frac{\partial S}{\partial \alpha_{n-1}^{(m)}} \tag{4.17}$$

($n-1$ equations) and condition $\sum_{i=0}^{n-1}\alpha_i = 1$. Expanding $S(\alpha)$ up to the terms of second order over $\alpha_i - \alpha_i^{(m)}$ on the plane $\sum_{i=0}^{n-1}\alpha_i = 1$ and calculating $(n-1)$-dimensional Gaussian integral over this plane, one obtains

$$A = \sqrt{D^{-1}\prod_{i=1}^{n-1}\frac{\alpha_{i-1}^{(m)}+\alpha_i^{(m)}}{\alpha_{i-1}^{(m)}\alpha_i^{(m)}}e^{kS(\alpha^{(m)})}} \tag{4.18}$$

where D is a determinant of a quadratic form $S(\alpha)$ on the plane near $\alpha^{(m)}$.

The derivatives $\partial S/\partial\alpha_i$ are equal to

$$\begin{aligned}
\frac{\partial S}{\partial\alpha_0} &= \log L_1\frac{\alpha_0+\alpha_1}{\alpha_0} \\
\frac{\partial S}{\partial\alpha_i} &= \log R_i L_{i+1}\frac{(\alpha_{i-1}+\alpha_i)(\alpha_i+\alpha_{i+1})}{\alpha_i^2} \qquad i=1,\ldots,n-2 \\
\frac{\partial S}{\partial\alpha_{n-1}} &= \log R_{n-1}L_n\frac{\alpha_{n-2}+\alpha_{n-1}}{\alpha_{n-1}}
\end{aligned} \tag{4.19}$$

$\alpha^{(m)}$ is easy to determine in the case $L_n = 1$ (it is clear that A in this case may only represent $\Re_{n-1}^{-1}\widetilde{W}_n$, because when $L_n = 1$, $W_{n+1} = 0$ for all t). The solution of (4.17) and (4.19) for $L_n = 1$ has the form

$$\alpha_i^{(m)} = \Delta_{i+1}\alpha_{i+1}^{(m)} \tag{4.20}$$

where

$$\Delta_i = L_i/R_i$$

so one has

$$\alpha_i^{(m)} = \alpha_{n-1}^{(m)}\prod_{j=i+1}^{n-1}\Delta_j. \tag{4.21}$$

It follows from (4.20) that the main contribution to the probabilities comes from trajectories satisfying

$$k_i^{(m)}R_{i+1} = k_{i+1}^{(m)}L_{i+1} \tag{4.22}$$

which is a detailed balance-type condition for the statistical system already described.

For solution (4.21) one has $\partial S/\partial\alpha_i = 0$, $i = 1,\ldots,n-1$, and, since $S(\alpha)$ has the property

$$S(\alpha) = \sum_{i=0}^{n-1}\frac{\partial S}{\partial\alpha_i}\alpha_i \tag{4.23}$$

we find that $S(\alpha^{(m)}) = 0$, and A is asymptotically independent of k (and t), which is expected since A is the occupation probability of point n on a closed ($L_n = 1$) lattice, and thus it should approach a constant at $t\to\infty$.

To determine this constant it is not necessary to actually compute the determinant D in (4.18). Instead, by using (4.21) we find that

$$k_i^{(m)} = k_{n-1}^{(m)} \prod_{j=i+1}^{n-1} \Delta_j . \tag{4.24}$$

Subsequently, during the time $t = n + 2\sum_{i=0}^{n-1} k_i^{(m)}$ the walker visited site i $k_{i-1}^{(m)} + k_i^{(m)}$ times, so in the limit $t \to \infty$ or $k \to \infty$ the occupation probability $\widetilde{W}_i^{(0)}$ at site i for the case $L_n = 1$ is

$$\begin{aligned}\widetilde{W}_i^{(0)} \underset{t\to\infty}{\longrightarrow} \frac{k_{i-1}^{(m)} + k_i^{(m)}}{2\sum_{i=0}^{n-1} k_i^{(m)}} &= \frac{k_{n-1}^{(m)}}{2\sum_{i=0}^{n-1} k_i^{(m)}} \left(\prod_{j=i}^{n-1} \Delta_j + \prod_{j=i+1}^{n-1} \Delta_j \right) \\ &= \frac{1}{2(1+\sum_{i=1}^{n-1}\prod_{j=i}^{n-1}\Delta_j)} \frac{1}{R_i} \prod_{j=i+1}^{n-1} \Delta_j \qquad i = 0, \ldots, n-1\end{aligned} \tag{4.25}$$

(for $i = n-1$ the last product should be changed to 1), while for $\widetilde{W}_n^{(0)}$ one has†

$$\widetilde{W}_n^{(0)} \underset{t\to\infty}{\longrightarrow} \frac{k_{n-1}^{(m)}}{2\sum_{i=0}^{n-1} k_i^{(m)}} = \frac{1}{2(1+\sum_{i=1}^{n-1}\prod_{j=i}^{n-1}\Delta_j)} . \tag{4.26}$$

Now by (4.10), (4.18) and (4.26)

$$\sqrt{D^{-1} \prod_{i=1}^{n-1} \frac{\alpha_{i-1}^{(m)} + \alpha_i^{(m)}}{\alpha_{i-1}^{(m)} \alpha_i^{(m)}}} = \frac{\Re_{n-1}^{-1}}{2(1+\sum_{i=1}^{n-1}\prod_{j=i}^{n-1}\Delta_j)} . \tag{4.27}$$

To evaluate $\widetilde{W}_n$ when $L_n \neq 1$, let us assume, as a first approximation valid for small R_n, that the values $\alpha_i^{(m)}$ do not change. Then at the point $\alpha = \alpha^{(m)}$ one gets

$$\begin{aligned}\frac{\partial S}{\partial \alpha_i} &= 0 \qquad i = 0, \ldots, n-2 \\ \frac{\partial S}{\partial \alpha_{n-1}} &= \log L_n\end{aligned} \tag{4.28}$$

while equation (4.27) holds true. This implies that the only difference in (4.18) between the case $L_n \neq 1$ and the previous case $L_n = 1$ is the factor e^{kS}, which by (4.23) and (4.28) is equal to

$$e^{kS} = e^{k\alpha_{n-1}^{(m)} \log L_n} = L_n^{k_{n-1}^{(m)}} \tag{4.29}$$

and by (4.10) and (4.18)

$$\widetilde{W}_n \bigg|_{t=n+2k} = \widetilde{W}_n^{(0)} L_n^{k_{n-1}^{(m)}} \tag{4.30}$$

† It is clear that the same formulae, (4.25) and (4.26), may be obtained as a solution of a stationary master equation $R_{i-1}\widetilde{W}_{i-1} - \widetilde{W}_i + L_{i+1}\widetilde{W}_{i+1} = 0$ with the boundary conditions $R_0 = L_n = 1$.

where $\widetilde{W}_n^{(0)}$ is the limiting value of occupation probability for $L_n = 1$, given by (4.26). This formula demonstrates that the reduction of occupation probability $\widetilde{W}_n$ in the case $L_n \neq 1$ in comparison with the case $L_n = 1$ occurs due to the possibility of escape from the region $0, \ldots, n$ from point n to the right, while the effective number of visits to the point n is the corresponding value in the most important path, i.e. $k_{n-1}^{(m)}$.

Including in (4.30) the expression for $\widetilde{W}_n^{(0)}$ and $k_{n-1}^{(m)}$ from (4.26) with $\sum_{i=0}^{n-1} k_i^{(m)} = k = \frac{1}{2}(t-n)$, one has an asymptotic relation

$$\widetilde{W}_n(t) = (1/\xi) L_n^{(t-n)/\xi} \tag{4.31}$$

where

$$\xi = 2\left(1 + \sum_{i=1}^{n-1} \prod_{j=i}^{n-1} \Delta_j\right). \tag{4.32}$$

Then, for $R_n \ll 1$ one has $L_n^{1/R_n} = (1 - R_n)^{1/R_n} \approx \mathrm{e}^{-1}$ and by (4.10) $W_{n+1}(t) = R_n \widetilde{W}_n(t-1)$, so from (4.31) we get

$$W_{n+1}(t) = (1/\zeta) \mathrm{e}^{-(t-n-1)/\zeta} \tag{4.33}$$

where $\zeta = \xi / R_n \approx \Delta_n \xi$, thus giving

$$\zeta = 2(\Delta_n + \Delta_n \Delta_{n-1} + \cdots + \Delta_n \Delta_{n-1} \cdots \Delta_1). \tag{4.34}$$

The value in parentheses is denoted in appendix A by κ_n' (see (A.6); for our case of the lattice without negative-numbered sites, κ_0' is equal to zero). From (A.1) one has now

$$\zeta = \overline{t_{n,n+1}} - 1 \tag{4.35}$$

where

$$\overline{t_{n,n+1}} = \frac{\mathrm{d}}{\mathrm{d}x}(r_n b_n)\bigg|_{x=1}$$

is the mean time of first passage from site n to site $n+1$.

Equations (4.33)–(4.35) provide the asymptotic form of the first-passage probability. In [10] a formulation, very different from that presented here, was proposed, and the asymptotic approximation analogous to (4.33) was obtained. This approximation gives, in our notation, the value $\overline{t_{n+1}}$ instead of ζ for the decay rate. However, this is not a contradiction, because in the case $R_n \ll 1$, considered here, ζ is the main part of $\overline{t_{n+1}}$, since it consists of all the terms of $\overline{t_{n+1}}$, proportional to R_n^{-1} (see (A.19), (A.13) and (A.6)). On the other hand our formulation does not need the assumption $n \gg 1$, used in [10].

5. On the problem of self-averaging

Here we want to further explore the interpretation, given in the previous section, of the generating function as some form of partition function. An important feature of large disordered systems is the self-averaging of free energy. Since the value $\tilde{p}_{n+1}$ plays the role of partition function, we expect that the corresponding 'free energy', $\log \tilde{p}_{n+1}$, is the self-averaging value, i.e. its fluctuations are small for large n. Let us show that this is true.

Since $\tilde{p}_{n+1} = \prod_1^n b_i$, we have

$$\log \tilde{p}_{n+1} = \sum_{i=1}^{n} \log b_i . \tag{5.1}$$

Then $\log b_i$ can be expanded into a Taylor series with respect to variables $\{c_j\}_{0\leqslant j\leqslant i-1}$:

$$\log b_i = \sum_{k_0,\ldots,k_{i-1}=0}^{\infty} z^{k_0+\cdots+k_{i-1}} c_0^{k_0} \ldots c_{i-1}^{k_{i-1}} M_{k_0,\ldots,k_{i-1}} \tag{5.2}$$

where

$$M_{k_0,\ldots,k_{i-1}} = \frac{1}{k_0!\cdots k_{i-1}!} \frac{\partial_0^k}{\partial c_0^{k_0}} \cdots \frac{\partial^{k_{i-1}}}{\partial c_{i-1}^{k_{i-1}}} \log b_i \bigg|_{\substack{z=1 \\ c_0=\cdots=c_{i-1}=0}} . \tag{5.3}$$

Using (3.2), coefficients M can be calculated like the coefficients N in the previous section resulting in

$$\log b_i = \sum_{j=1}^{i} \sum_{k_{i-1},\ldots,k_{i-j}=1}^{\infty} z^{k_{i-j}+\cdots+k_{i-1}} c_{i-j}^{k_{i-j}} \ldots c_{i-1}^{k_{i-1}} \cdot \frac{1}{k_{i-1}} \cdot \prod_{m=i-j}^{i-2} \binom{k_m + k_{m+1} - 1}{k_m} \tag{5.4}$$

where for $j = 1$ the last product should be changed to 1.

In this formula, contrary to (4.2), all k_m are larger than or equal to 1, so that (5.1) and (5.4) express $\log \tilde{p}_{n+1}$ as a sum over 'connected clusters' of all lengths $j \geqslant 1$. It may be shown, then, using the estimates of the previous section, that the contribution of the jth term to $\log b_i$ in (5.4) exponentially decreases with j, so that only small clusters with length not increasing with n, contribute to $\log \tilde{p}_{n+1}$. Due to the central limit theorem, this leads us to the conclusion that for large n, $\log \tilde{p}_{n+1}$ and consequently $\log p_{n+1}$ are self-averaging†.

Let us now rewrite (2.2) in the form

$$\begin{aligned} W_n(t) &= \frac{1}{(t-n)!} \frac{\mathrm{d}^{t-n}}{\mathrm{d}x^{t-n}} (x^{-n} p_n(x)) \bigg|_{x=0} \\ &= \frac{1}{(t-n)!} \frac{\mathrm{d}^{t-n}}{\mathrm{d}x^{t-n}} (\Re_{n-1} \tilde{p}_n(x)) \bigg|_{x=0} \end{aligned} \tag{5.5}$$

† The exponential decrease of the jth term in (5.4) with j is evident without estimates in the limit $x \to 0$ which is important for the calculation of probabilities (see (2.2)). In the opposite limit $x \to 1$ which is important for the calculation of moments (see (2.3)), the self-averaging of $\log \tilde{p}_{n+1}$ follows from (5.1) and the relation $b_k|_{x=1} = 1/R_k$, following from (2.4) or (2.7).

and along with $\overline{t_n^i}$, $W_n(t)$, given by (2.3), (5.5), introduce corresponding 'cumulant' values $\langle\langle\overline{t_n^i}\rangle\rangle$, $\langle\langle W_n(t)\rangle\rangle$ by

$$\langle\langle\overline{t_n^i}\rangle\rangle = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^i \log p_n(x)\bigg|_{x=1} \tag{5.6}$$

$$\langle\langle W_n(t)\rangle\rangle = \frac{1}{(t-n)!}\frac{\mathrm{d}^{t-n}}{\mathrm{d}x^{t-n}}\log(\Re_{n-1}\tilde{p}_n(x))\bigg|_{x=0} \tag{5.7}$$

Then, since $\log(\Re_{n-1}\tilde{p}_n(x))$ is also self-averaging together with $\log\tilde{p}_n(x)$, we may state that these are 'cumulant' values (5.6) and (5.7), rather than the usual (2.3) and (5.5), which are self-averaging.

One can express $\langle\langle\overline{t_n^i}\rangle\rangle$ and $\langle\langle W_n(t)\rangle\rangle$ through $\{\overline{t_n^{i'}}\}_{1\leqslant i'\leqslant i}$ and $\{W_n(t')\}_{n\leqslant t'\leqslant t}$ correspondingly using the well known formulae for cumulants (see, e.g. [14]), following from the definitions (2.3), (5.5), (5.6) and (5.7). Actual calculations are simplified by the relation $p_n(x)|_{x=1} = \tilde{p}_n(x)|_{x=0} = 1$†. This relation also implies that self-averaging of $\log p_n$ and $\log\tilde{p}_n$ cannot be considered as a trivial effect of damping by the logarithmic function, of the fluctuations of its argument.

6. Summary

We have presented a method for calculating the properties of a discrete-time random walk on the one-dimensional random lattice. The method starts with a system of equations (2.4) and (2.5) for generating functions. It allows the explicit solution for the time dependence of the first-passage and occupation probabilities in section 3 to be obtained. These solutions, however, were obtained for a particular realization of the random variables and it is not clear how to average them over different realizations. To this end, the representation, obtained in section 4, seems to be more useful. This representation separates the 'physical' aspect of the random walk, described by the values L_k, from its 'geometrical' aspect, described by the constants $N_{k_0,\ldots,k_{n-1}}$. While the values L_k may depend on the particular physical situation, the constants $N_{k_0,\ldots,k_{n-1}}$ describe the inner geometrical properties of the one-dimensional lattice and do not change from one system to another. Since the expressions, obtained in section 4, are all polynomial in $\{L_k\}$, they are convenient for an averaging procedure. The 'partition-function' representation may also provide a basis for an investigation of the effects of correlated disorder. Since this representation has the form of the sum over configurations, it allows the use of standard field-theoretical methods, as was illustrated by the simple saddle-point evaluation of probabilities. We also found a class of 'cumulant' probabilities and moments which are self-averaging in the thermodynamic limit.

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† We rewrote (2.2) in the form (5.5) because $\tilde{p}_n(x)|_{x=0} = 1$, while $p_n(x)|_{x=0} = 0$.

Appendix A. The moments of first-passage time

In this appendix we shall find two first moments of first-passage time. Our consideration will be constrained to the condition (satisfied in most practically interesting cases) that the full probability that a random walker achieves point $n > 0$ at any time, which is, by (2.1), $p_n(1)$, is equal to 1, or, in other words, that the probability of going to $-\infty$ without visiting point $n > 0$ is equal to 0. This condition is, in fact, independent of n and is satisfied unless $L_k, k \to -\infty$ is abnormally large (in particular, it is satisfied in the important case of a half-infinite lattice, cut at some $n_0 \leqslant 0$).

Under this condition it is easy to see from (2.4) or (2.7) that $b_k|_{x=1} = 1/R_k$, and it happens to be convenient to express b_k to order y^2, where $y = 1 - x$, in the form

$$b_k = (1/R_k)(1 - 2y\kappa'_k + y^2\kappa''_k) \tag{A.1}$$

where κ'_k and κ''_k are independent of y.

Including this expression in (2.7) and comparing the terms of the same order of y, one has for every $n \geqslant 1$

$$\kappa'_n = \Delta_n + \Delta_n \kappa'_{n-1} \tag{A.2}$$

where

$$\Delta_n = L_n/R_n \tag{A.3}$$

and

$$\kappa''_n = A_n + \Delta_n \kappa''_{n-1} \tag{A.4}$$

where

$$\begin{aligned} A_n &= 4\kappa'^2_n + \Delta_n + 4\Delta_n \kappa'_{n-1} \\ &= 4\kappa'^2_n + 4\kappa'_n - 3\Delta_n \end{aligned} \tag{A.5}$$

(for the last transformation in (A.5) we used (A.2)).

By iterating (A.2) one has for $n \geqslant 1$

$$\begin{aligned} \kappa'_n &= \Delta_n + \Delta_n(\Delta_{n-1} + \Delta_{n-1}(\cdots + \Delta_2(\Delta_1 + \Delta_1\kappa'_0)\cdots)) \\ &= \Delta_n + \Delta_n\Delta_{n-1} + \cdots + \Delta_n\Delta_{n-1}\ldots\Delta_1 + \Delta_n\Delta_{n-1}\ldots\Delta_1\kappa'_0 \end{aligned} \tag{A.6}$$

where κ'_0 in turn may be expressed through $\{\Delta_k\}_{k\leqslant 0}$, but we prefer to leave it explicit, because it can be expressed through the straightforwardly observable value $\overline{t_1}$ (see later). In the important particular case, when the lattice is cut in point $n_0 = 0$ and has no points with $n < 0$, one has $L_0 = 0$, $R_0 = 1$, $b_0(x) = 1$ and $\kappa'_0 = \kappa''_0 = 0$.

By iterating (A.4) one has

$$\begin{aligned} \kappa''_n &= A_n + \Delta_n(A_{n-1} + \Delta_{n-1}(A_{n-2} + \cdots + \Delta_2(A_1 + \Delta_1\kappa''_0)\cdots)) \\ &= A_n + \Delta_n A_{n-1} + \Delta_n\Delta_{n-1}A_{n-2} \\ &\quad + \cdots + \Delta_n\Delta_{n-1}\ldots\Delta_2 A_1 + \Delta_n\Delta_{n-1}\ldots\Delta_1\kappa''_0 \end{aligned} \tag{A.7}$$

or, by using (A.5),

$$\kappa_n'' = 4\sum_{k=1}^{n}\Delta_n\dots\Delta_{k+1}(\kappa_k'^2+\kappa_k') - 3\sum_{k=1}^{n}\Delta_n\dots\Delta_k + \Delta_n\dots\Delta_1\kappa_0'' \tag{A.8}$$

where for $k=n$ the product $\Delta_n\dots\Delta_{k+1}$ in the first term should be replaced by 1.

The first term in right-hand side of (A.8) may be transformed using the identity

$$\Delta_n\dots\Delta_{k+1}\kappa_k' = \kappa_n' - \Delta_n - \Delta_n\Delta_{n-1} - \dots - \Delta_n\Delta_{n-1}\dots\Delta_{k+1} \qquad k=0,\dots,n-1 \tag{A.9}$$

from which it is follows that

$$\Delta_n\dots\Delta_{k+1}(\kappa_k'^2+\kappa_k') = (\kappa_n' - \Delta_n - \Delta_n\Delta_{n-1} - \dots - \Delta_n\dots\Delta_{k+2})\kappa_k' \qquad k=0,\dots,n-2 \tag{A.10}$$

and

$$\Delta_n(\kappa_{n-1}'^2+\kappa_{n-1}') = \kappa_n'\kappa_{n-1}'. \tag{A.11}$$

From (A.8)–(A.11) one has, after some algebra,

$$\kappa_n'' = 4\kappa_n' f_n + \kappa_n' - 4\sum_{k=1}^{n-2}\Delta_n\dots\Delta_{k+2}f_k + 3\Delta_n\dots\Delta_1\kappa_0' + \Delta_n\dots\Delta_1\kappa_0'' \qquad n=1,2,\dots \tag{A.12}$$

where for $n=1,2$ the sum in the right-hand side must be dropped and we introduced the notation

$$f_k = \sum_{i=1}^{k}\kappa_i' = \varphi_k + \kappa_{k,1}\kappa_0' \qquad k=1,2,\dots \tag{A.13}$$

$$\varphi_k = \sum_{i=1}^{k}\Delta_i + \sum_{i=1}^{k-1}\Delta_i\Delta_{i+1} + \dots + \Delta_1\dots\Delta_k \qquad k=1,2,\dots \tag{A.14}$$

$$\kappa_{k,j} = \sum_{i=j}^{k}\Delta_i\dots\Delta_j \qquad k\geqslant j. \tag{A.15}$$

In (A.12) κ_0'', like κ_0', may be expressed as a function of $\{\Delta_k\}_{k\leqslant 0}$ or as a function of the observables $\overline{t_1}$ and $\overline{t_1^2}$ (see later).

Putting (A.1), (A.6), (A.12) to (2.6) and leaving only the terms of order y and y^2, one has for p_{n+1} :

$$\begin{aligned} p_{n+1} &= \prod_{k=0}^{n} b_k r_k = (1-y)^{n+1}\prod_{k=0}^{n}(1-2y\kappa_k' + y^2\kappa_k'') \\ &= \left(1-(n+1)y+\frac{n(n+1)}{2}y^2\right) \\ &\quad\times\left(1-2y(f_n+\kappa_0') + y^2\left(4\sum_{k=2}^{n}\kappa_k' f_{k-1} + 4f_n\kappa_0' + \sum_{k=0}^{n}\kappa_k''\right)\right) + \mathrm{O}(y^3). \end{aligned} \tag{A.16}$$

It is convinient to make the following transformations in the right-hand side of (A.16):

$$\begin{aligned}
4\sum_{k=2}^{n}\kappa'_k f_{k-1}+\sum_{k=1}^{n}\kappa''_k
&= 8\sum_{k=1}^{n}\kappa'_k f_k-4\sum_{k=1}^{n}\kappa'^2_k+(3\kappa'_0+\kappa''_0)\kappa_{n,1}+f_n\\
&\quad -4\sum_{i=3}^{n}\sum_{k=1}^{i-2}\Delta_i\ldots\Delta_{k+2}f_k
\end{aligned}\tag{A.17}$$

where we used the identity $f_{k-1}=f_k-\kappa'_k$ and (A.12) for κ''_k. Now putting $\kappa'_k=f_k-f_{k-1}$, one has for the right-hand side of (A.17)

$$4f_n^2+f_n-4\sum_{k=1}^{n-2}\kappa_{n,k+2}f_k+(3\kappa'_0+\kappa''_0)\kappa_{n,1}$$

where for $n=1,2$ the sum in this expression as well as in the right-hand side of (A.18), (A.20) and (A.21) must be dropped.

Now we have for p_{n+1} up to the order of y^2

$$\begin{aligned}
p_{n+1}=\left(1-(n+1)y+\frac{n(n+1)}{2}y^2\right)\bigg(1-2y(f_n+\kappa'_0)+y^2\bigg(4f_n^2+f_n\\
-4\sum_{k=1}^{n-2}\kappa_{n,k+2}f_k+4f_n\kappa'_0+(3\kappa'_0+\kappa''_0)\kappa_{n,1}+\kappa''_0\bigg)\bigg)
\end{aligned}\tag{A.18}$$

so by (2.3)

$$\begin{aligned}
\overline{t_{n+1}}&=n+1+2(f_n+\kappa'_0)\\
&=n+1+2\varphi_n+2\kappa'_0(1+\kappa_{n,1})
\end{aligned}\tag{A.19}$$

and

$$\begin{aligned}
\overline{t^2_{n+1}}&=(n+1)^2+4(n+2)f_n+8f_n^2-8\sum_{k=1}^{n-2}\kappa_{n,k+2}f_k\\
&\quad+4(n+\tfrac{3}{2})\kappa'_0+8f_n\kappa'_0+2(3\kappa'_0+\kappa''_0)\kappa_{n,1}+2\kappa''_0.
\end{aligned}\tag{A.20}$$

From (A.19) and (A.20) it is an easy matter to obtain an expression for the dispersion:

$$\overline{t^2_{n+1}}-\overline{t_{n+1}}^2=4f_n+4f_n^2-8\sum_{k=1}^{n-2}\kappa_{n,k+2}f_k+2\kappa'_0-4\kappa'^2_0+2(3\kappa'_0+\kappa''_0)\kappa_{n,1}+2\kappa''_0.\tag{A.21}$$

By (A.19), (A.20) κ_0' and κ_0'' may be expressed in terms of the observables $\overline{t_1}$ and $\overline{t_1^2}$:

$$\kappa_0' = \tfrac{1}{2}(\overline{t_1} - 1) \tag{A.22}$$

$$\kappa_0'' = \tfrac{1}{2}(\overline{t_1^2} - 1 - 6\kappa_0') = \tfrac{1}{2}(\overline{t_1^2} - 3\overline{t_1} + 2). \tag{A.23}$$

In the case where the $\{L_k\}$ are independent random variables with the same distribution for all k, equation (A.19) for $\overline{t_{n+1}}$ may be easily averaged over configurations to yield

$$\langle\overline{t_{n+1}}\rangle = \frac{2\Delta^{n+2} + (n+1)(1-\Delta^2) - 2\Delta}{(\Delta-1)^2} + 2\langle\kappa_0'\rangle\frac{\Delta^{n+1}-1}{\Delta-1} \tag{A.24}$$

where $\Delta = \langle\Delta_k\rangle$ and angular brackets $\langle\rangle$ denote configurational averaging.

If the lattice has no sites with $n < 0$, then $\overline{t_1} = \overline{t_1^2} = 1$ and by (A.22) and (A.23) $\kappa_0' = \kappa_0'' = 0$ as was already mentioned. In this case (A.19) and (A.24) coincide with the results of [7–9] (see also [6]).

Appendix B. More cases allowing explicit representation of the generating function

Explicit expressions for the generating function, analogous to (3.10) and (3.13), may be obtained not only for first passage from 0 to $n+1$, but also for some other quantities. First, if the lattice has the finite length $N \geqslant n+1$, then the generating function for the occupation probabilities may be obtained. This function is equal to

$$p_{n+1}^{\mathrm{occ}} = p_{n+1}B_{n+1} \tag{B.1}$$

where B_{n+1} is the sum over all paths, coming from site $n+1$ back to $n+1$. It satisfies the equation

$$B_{n+1} = 1 + (l_{n+1}b_n r_n + r_{n+1}\tilde{b}_{n+2}l_{n+2})B_{n+1} \tag{B.2}$$

where $\tilde{b}_{n+2}$ is the sum over all paths from $n+2$ back to $n+2$ 'through the right'. For arbitrary k, $\tilde{b}_k$ expressed through $\{L_i\}_{k\leqslant i\leqslant N}$ in explicit form by the same formulae as b_k through $\{R_i\}_{0\leqslant i\leqslant k}$ with the obvious interchange $R_i \to L_{N-i}$. From (B.1) and (B.2) one gets

$$p_{n+1}^{\mathrm{occ}} = p_{n+1}\frac{1}{1 - l_{n+1}r_n b_n - r_{n+1}l_{n+2}\tilde{b}_{n+2}}. \tag{B.3}$$

If $n+1$ is the last site ($n+1 = N$), then $r_{n+1} = 0$ and (B.3) goes to

$$\begin{aligned} p_{n+1}^{\mathrm{endocc}} &= p_{n+1}b_{n+1} \\ &= x^{n+1}\Re_n\frac{1}{\sum_{m=0}^{m_{n+1}} b_{n+1}^{(m)} z^m} \\ &= \Re_n\sum_{m=0}^{\infty} d_{n+1}^{(m)} x^{2m+n+1} \end{aligned} \tag{B.4}$$

and corresponding probability is

$$W_{n+1}^{\text{endocc}}\Big|_{t=n+1+2m} = \Re_n d_{n+1}^{(m)}. \tag{B.5}$$

Second, if the walker started not from 0, but from some site i, then instead of $\prod_{k=0}^{n} b_k r_k$ one gets for the generating function of the probability of first passage from site i to site $n+1$, $p_{n+1}^{(i)}$:

$$\begin{aligned} p_{n+1}^{(i)} &= \prod_{k=i}^{n} b_k r_k \\ &= x^{n-i+1} \frac{D_{i-1}}{D_n} \prod_{k=i}^{n} R_k \end{aligned} \tag{B.6}$$

where we used (3.3) and relation $N_i = D_{i-1}$. Similarly, the generating function of occupation probability for this initial condition is still given by (B.3) and (B.4) if we replace p_{n+1} by $p_{n+1}^{(i)}$ there.

Finally, on the infinite lattice these formulae also hold, provided the finite sums are convergent when extended to infinity. In conclusion we may state that explicit expressions for generating functions (and, consequently, for the corresponding probabilities) may be obtained by this approach for a number of different cases.

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